

SERRE WEIGHTS FOR $U(n)$.

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ABSTRACT. We study the weight part of (a generalisation of) Serre's conjecture for mod l Galois representations associated to automorphic representations on unitary groups of rank n for odd primes l . Given a modular Galois representation, we use automorphy lifting theorems to prove that it is modular in many other weights. We make no assumptions on the ramification or inertial degrees of l . We give an explicit strengthened result when $n = 3$ and l splits completely in the underlying CM field.

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1. INTRODUCTION

In recent years there has been considerable progress in formulating generalisations of Serre's conjecture, and in particular of the weight part of Serre's conjecture, for higher-dimensional groups; cf. [ADP02], [Her09], [Gee11], [EGHS14]. There has been rather less progress in proving cases of these conjectures; indeed, the only results that we are aware of are the essentially complete treatment of the ordinary case for definite unitary groups in [GG12], and the results of [EGH13] for definite unitary groups of rank 3.

In the present paper, we use the automorphy lifting theorems developed in [BLGG11], [BLGG12] and [BLGGT14b] to prove that a modular Galois representation, coming from an automorphic form on $U(n)$, is necessarily modular in a number of additional weights predicted by the conjectures of [Her09] and [EGHS14]. Rather complete results are available in the case $n = 2$, which are worked out in detail in the papers [BLGG13, GLS14, GLS13], so we concentrate in this paper on the case that $n > 2$. The additional complications are twofold. Firstly, we no longer know that any modular Galois representation admits a potentially diagonalizable lift (in the case $n = 2$, this is proved in [BLGG13] as a consequence of the

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results of [Kis09] and [Gee06]). Secondly, the relationship between being modular of some weight and having an automorphic lift of some weight is substantially more complicated for $n > 2$ than it is for $n = 2$; in particular, it is no longer the case that given an irreducible mod l representation F of $\mathrm{GL}_n(\mathbb{F}_l)$, there is necessarily an irreducible characteristic zero algebraic representation W of GL_n whose reduction modulo l is F . Instead, one finds that F is the socle of the reduction modulo l of some W , and this gives strictly weaker information.

As a result of these two difficulties, our main theorems have two restrictions. Let F be a CM field, and let $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be our given modular Galois representation. Firstly, we must assume that \bar{r} has a potentially diagonalizable automorphic lift. This assumption is perhaps not as serious as it initially sounds, as it is conjecturally always satisfied, and in particular is known to hold provided that l is unramified in F and \bar{r} has an automorphic lift of sufficiently small weight. Secondly, rather than prove that \bar{r} is modular of some particular weight, we typically only provide a list of weights, and guarantee that \bar{r} is modular of some weight in this list. In fact, it is often the case that only one weight on this list is predicted by the conjectures of [Her09] and [EGHS14], and it should presumably be possible to prove modularity in this weight in many cases using integral p -adic Hodge theory. We carry out such an analysis in detail in the case $n = 3$, defining a list of conjectural weights $W^{\mathrm{explicit}}(\bar{r})$, and obtaining the following result (Theorem 5.1.4).

Theorem A. *Let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, and that l splits completely in F . Suppose that $l > 2$, and that $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_l)$ is an irreducible representation with split ramification. Assume that there is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_F)$ of weight $\mu \in (\mathbb{Z}_+^3)_0^{\mathrm{Hom}(F, \mathbb{C})}$ and level prime to l such that*

- $\bar{r} \cong \bar{r}_{l, \iota}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{-2}$).
- For each $\tau \in \mathrm{Hom}(F, \mathbb{C})$, $\mu_{\tau, 1} - \mu_{\tau, 3} \leq l - 3$.
- $\bar{r}(G_{F(\zeta_l)})$ is adequate.

Let $a \in (\mathbb{Z}_+^3)_{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}^{\mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ be a generic Serre weight. Assume that $a \in W^{\mathrm{explicit}}(\bar{r})$. Then \bar{r} is modular of weight a .

(See sections 2 and 4 for any unfamiliar terminology, and section 5 for the definition of “generic” that we are using, which is extremely mild.) We should point out that we do *not* expect that $W^{\mathrm{explicit}}(\bar{r})$ contains all the weights in which \bar{r} is modular; rather, it consists of those weights which are “obvious” in the terminology of [EGHS14]. (It is perhaps worth remarking that despite the name, it is not obvious that \bar{r} is modular in any of these weights!) In order to prove this theorem we make use of Fontaine-Laffaille theory; it seems likely that if one could compute the possible reductions of crystalline Galois representations outside of the Fontaine-Laffaille range then one could prove an analogous theorem for $n > 3$.

We now outline the structure of this paper. In Section 2 we define the spaces of automorphic forms that we work with, and define what it means for \bar{r} to be modular of some weight. In Section 3 we establish the main lifting theorem that we need, a corollary of the results of [BLGGT14b]. In Section 4 we define the set of weights $W^{\mathrm{explicit}}(\bar{r})$, recall some results from Fontaine-Laffaille theory, and prove our main results for arbitrary n . Finally, in Section 5 we prove Theorem A.

1.1. Notation. If M is a field, we let G_M denote its absolute Galois group. We write all matrix transposes on the left; so ${}^t A$ is the transpose of A . Let ϵ_l denote the l -adic cyclotomic character, and $\bar{\epsilon}_l$ or ω_l the mod l cyclotomic character. If M is a finite extension of \mathbb{Q}_p for some p , we write I_M for the inertia subgroup of G_M . If R is a local ring we write \mathfrak{m}_R for the maximal ideal of R .

We fix an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} . For each prime p we fix an algebraic closure $\bar{\mathbb{Q}}_p$ of \mathbb{Q}_p , and we fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

If W is a de Rham representation of G_K over $\bar{\mathbb{Q}}_l$ and if $\tau : K \hookrightarrow \bar{\mathbb{Q}}_l$ then by definition the multiset $\text{HT}_\tau(W)$ of Hodge-Tate weights of W with respect to τ contains i with multiplicity $\dim_{\bar{\mathbb{Q}}_l}(W \otimes_{\tau, K} \widehat{K}(i))^{G_K}$. Thus for example $\text{HT}_\tau(\epsilon_l) = \{-1\}$.

If K is a finite extension of \mathbb{Q}_p for some p , we will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $\text{GL}_n(K)$, then $\text{rec}_K(\pi)$ is a Weil-Deligne representation of the Weil group W_K . We will write rec for rec_K when the choice of K is clear. We write $\text{Art}_K : K^\times \rightarrow W_K$ for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements.

Let K be a finite extension of \mathbb{Q}_l with residue field k . For each $\sigma \in \text{Hom}(k, \bar{\mathbb{F}}_l)$ we define the fundamental character ω_σ corresponding to σ to be the composite

$$I_{K^{\text{ab}}/K} \xrightarrow{\text{Art}_K^{-1}} \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma^{-1}} \bar{\mathbb{F}}_l^\times.$$

For any algebraic extension L of \mathbb{Q}_l , we often denote by $\text{Hom}(K, L)$ the set of field homomorphisms from K to L which are continuous for the l -adic topologies on K and L (or equivalently, which are \mathbb{Q}_l -linear).

2. DEFINITIONS

2.1. Let l be a prime, and let F be an imaginary CM field with maximal totally real field subfield F^+ . We assume throughout this paper that:

- F/F^+ is unramified at all finite places.
- Every place $v|l$ of F^+ splits in F .
- If n is even, then $n[F^+ : \mathbb{Q}]/2$ is also even.

Under these hypotheses, there is a reductive algebraic group G/F^+ with the following properties:

- G is an outer form of GL_n , with $G/F \cong \text{GL}_{n/F}$.
- If v is a finite place of F^+ , G is quasi-split at v .
- If v is an infinite place of F^+ , then $G(F_v^+) \cong U_n(\mathbb{R})$.

To see that such a group exists, one may argue as follows. Let B denote the matrix algebra $M_n(F)$. An involution \dagger of the second kind on B gives a reductive group G_\dagger over F^+ by setting

$$G_\dagger(R) = \{g \in B \otimes_{F^+} R : g^\dagger g = 1\}$$

for any F^+ -algebra R . Any such G_\dagger is an outer form of GL_n , with $G_\dagger/F \cong \text{GL}_{n/F}$. One can choose \dagger such that

- If v is a finite place of F^+ , G_\dagger is quasi-split at v .
- If v is an infinite place of F^+ , then $G_\dagger(F_v^+) \cong U_n(\mathbb{R})$.

To see this, one uses the argument of Lemma I.7.1 of [HT01]. We then fix some choice of \ddagger as above, and take $G = G_{\ddagger}$.

As in section 3.3 of [CHT08] we define a model for G over \mathcal{O}_{F^+} in the following way. We choose an order \mathcal{O}_B in B such that $\mathcal{O}_B^{\ddagger} = \mathcal{O}_B$, and $\mathcal{O}_{B,w}$ is a maximal order in B_w for all places w of F which are split over F^+ (see section 3.3 of [CHT08] for a proof that such an order exists). Then we can define G over \mathcal{O}_{F^+} by setting

$$G(R) = \{g \in \mathcal{O}_B \otimes_{\mathcal{O}_{F^+}} R : g^{\ddagger}g = 1\}$$

for any \mathcal{O}_{F^+} -algebra R .

If v is a place of F^+ which splits as ww^c over F , then we choose an isomorphism

$$\iota_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{F,v}) = M_n(\mathcal{O}_{F_w}) \oplus M_n(\mathcal{O}_{F_{w^c}})$$

such that $\iota_v(x^{\ddagger}) = {}^t\iota_v(x)^c$. This gives rise to an isomorphism

$$\iota_w : G(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_w})$$

sending $\iota_v^{-1}(x, {}^t x^{-c})$ to x .

Let K be an algebraic extension of \mathbb{Q}_l in $\overline{\mathbb{Q}_l}$ which contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}_l}$, let \mathcal{O} denote the ring of integers of K , and let k denote the residue field of K . Let S_l denote the set of places of F^+ lying over l , and for each $v \in S_l$ fix a place \tilde{v} of F lying over v . Let \tilde{S}_l denote the set of places \tilde{v} for $v \in S_l$.

Let W be an \mathcal{O} -module with an action of $G(\mathcal{O}_{F^+,l})$, and let $U \subset G(\mathbb{A}_{F^+}^{\infty})$ be a compact open subgroup with the property that for each $u \in U$, if u_l denotes the projection of u to $G(F_l^+)$, then $u_l \in G(\mathcal{O}_{F_l^+})$. Let $S(U, W)$ denote the space of algebraic modular forms on G of level U and weight W , i.e. the space of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty}) \rightarrow W$$

with $f(gu) = u_l^{-1}f(g)$ for all $u \in U$.

Let \tilde{I}_l denote the set of embeddings $F \hookrightarrow K$ giving rise to a place in \tilde{S}_l . For any $\tilde{v} \in \tilde{S}_l$, let $\tilde{I}_{\tilde{v}}$ denote the set of elements of \tilde{I}_l lying over \tilde{v} . Let \mathbb{Z}_+^n denote the set of tuples $(\lambda_1, \dots, \lambda_n)$ of integers with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. For any $\lambda \in \mathbb{Z}_+^n$, view λ as a dominant character of the algebraic group $\mathrm{GL}_{n/\mathcal{O}}$ in the usual way, and let M'_{λ} be the algebraic \mathcal{O} -representation of GL_n given by

$$M'_{\lambda} := \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/\mathcal{O}}$$

where B_n is the standard Borel subgroup of GL_n , and w_0 is the longest element of the Weyl group (see [Jan03] for more details of these notions). Write M_{λ} for the \mathcal{O} -representation of $\mathrm{GL}_n(\mathcal{O})$ obtained by evaluating M'_{λ} on \mathcal{O} . For any $\lambda \in (\mathbb{Z}_+^n)^{\tilde{I}_{\tilde{v}}}$, let W_{λ} be the free \mathcal{O} -module with an action of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ given by

$$W_{\lambda} := \otimes_{\tau \in \tilde{I}_{\tilde{v}}} M_{\lambda_{\tau}} \otimes_{\mathcal{O}_{F_{\tilde{v}}}, \tau} \mathcal{O}.$$

We give this an action of $G(\mathcal{O}_{F^+,v})$ via $\iota_{\tilde{v}}$. For any $\lambda \in (\mathbb{Z}_+^n)^{\tilde{I}_l}$, let W_{λ} be the free \mathcal{O} -module with an action of $G(\mathcal{O}_{F^+,l})$ given by

$$W_{\lambda} := \otimes_{\tilde{v} \in \tilde{S}_l} W_{\lambda_{\tilde{v}}}.$$

If A is an \mathcal{O} -module we let

$$S_{\lambda}(U, A) := S(U, W_{\lambda} \otimes_{\mathcal{O}} A).$$

For any compact open subgroup U as above of $G(\mathbb{A}_{F^+}^\infty)$ we may write $G(\mathbb{A}_{F^+}^\infty) = \coprod_i G(F^+)t_i U$ for some finite set $\{t_i\}$. Then there is an isomorphism

$$S(U, W) \rightarrow \oplus_i W^{U \cap t_i^{-1} G(F^+) t_i}$$

given by $f \mapsto (f(t_i))_i$. We say that U is *sufficiently small* if for some finite place v of F^+ the projection of U to $G(F_v^+)$ contains no element of finite order other than the identity. Suppose that U is sufficiently small. Then for each i as above we have $U \cap t_i^{-1} G(F^+) t_i = \{1\}$, so taking $W = W_\lambda \otimes_{\mathcal{O}} A$ we see that for any \mathcal{O} -module A , we have

$$S_\lambda(U, A) \cong S_\lambda(U, \mathcal{O}) \otimes_{\mathcal{O}} A.$$

We note when U is not sufficiently small, we still have $S_\lambda(U, A) \cong S_\lambda(U, \mathcal{O}) \otimes_{\mathcal{O}} A$ whenever A is \mathcal{O} -flat.

We now recall the relationship between our spaces of algebraic automorphic forms and the space of automorphic forms on G . Write $S_\lambda(\overline{\mathbb{Q}}_l)$ for the direct limit of the spaces $S_\lambda(U, \overline{\mathbb{Q}}_l)$ over compact open subgroups U as above (with the transition maps being the obvious inclusions $S_\lambda(U, \overline{\mathbb{Q}}_l) \subset S_\lambda(V, \overline{\mathbb{Q}}_l)$ whenever $V \subset U$). Concretely, $S_\lambda(\overline{\mathbb{Q}}_l)$ is the set of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}) \rightarrow W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$$

such that there is a compact open subgroup U of $G(\mathbb{A}_{F^+}^{\infty, l}) \times G(\mathcal{O}_{F^+, l})$ with

$$f(gu) = u_l^{-1} f(g)$$

for all $u \in U$, $g \in G(\mathbb{A}_{F^+})$. This space has a natural left action of $G(\mathbb{A}_{F^+}^\infty)$ via

$$(g \cdot f)(h) := g_l f(hg).$$

Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$, there is a unique embedding $\tilde{\tau} : F \hookrightarrow \mathbb{C}$ extending τ such that $\iota^{-1} \tilde{\tau} \in \tilde{I}_l$. Let σ_λ denote the representation of $G(F_\infty^+)$ given by $W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l \otimes_{\overline{\mathbb{Q}}_l, \iota} \mathbb{C}$, with an element $g \in G(F_\infty^+)$ acting via $\otimes_\tau \tilde{\tau}(\iota_\tau(g))$. Let \mathcal{A} denote the space of automorphic forms on $G(F^+) \backslash G(\mathbb{A}_{F^+})$. From the proof of Proposition 3.3.2 of [CHT08], one easily obtains the following.

Lemma 2.1.1. *There is an isomorphism of $G(\mathbb{A}_{F^+}^\infty)$ -modules*

$$S_\lambda(\overline{\mathbb{Q}}_l) \xrightarrow{\sim} \text{Hom}_{G(F_\infty^+)}(\sigma_\lambda^\vee, \mathcal{A}).$$

In particular, we note that $S_\lambda(\overline{\mathbb{Q}}_l)$ is a semi-simple admissible $G(\mathbb{A}_{F^+}^\infty)$ -module.

Following [CHT08], we say that a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ is RACSDC (regular, algebraic, conjugate self dual, and cuspidal) if

- π_∞ has the same infinitesimal character as some irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}} \text{GL}_n$, and
- $\pi^c \cong \pi^\vee$.

We say that π has level prime to l if π_v is unramified for all $v|l$. If Ω is an algebraically closed field of characteristic 0 we write $(\mathbb{Z}_+^n)_0^{\text{Hom}(F, \Omega)}$ for the subset of elements $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, \Omega)}$ such that

$$\lambda_{\tau, i} + \lambda_{\tau \circ c, n+1-i} = 0$$

for all τ, i .

If $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, \mathbb{C})}$ we write Σ_λ for the irreducible algebraic representation of $\text{GL}_n^{\text{Hom}(F, \mathbb{C})}$ given by the tensor product over τ of the irreducible representations with highest weights λ_τ . We say that a RACSDC automorphic representation π of $\text{GL}_n(\mathbb{A}_F)$ has weight $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, \mathbb{C})}$ if π_∞ has the same infinitesimal character as Σ_λ^\vee . If this is the case then necessarily $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \mathbb{C})}$.

Theorem 2.1.2. *If π is a RACSDC automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ of weight λ , then there is a continuous semisimple representation*

$$r_{l,\iota}(\pi) : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_l)$$

such that

- (1) $r_{l,\iota}(\pi)^c \cong r_{l,\iota}(\pi)^\vee \otimes \epsilon_l^{1-n}$.
- (2) The representation $r_{l,\iota}(\pi)$ is de Rham, and is crystalline if π has level prime to l . If $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ then

$$\text{HT}_\tau(r_{l,\iota}(\pi)) = \{\lambda_{\iota\tau,1} + n - 1, \dots, \lambda_{\iota\tau,n}\}.$$

- (3) For each finite place v of l , we have

$$\iota \text{WD}(r_{l,\iota}(\pi)|_{G_{F_v}})^{\text{F-ss}} \cong \text{rec}(\pi_v^\vee \otimes |\det|^{(1-n)/2}).$$

Here $\text{WD}(r_{l,\iota}(\pi)|_{G_{F_v}})^{\text{F-ss}}$ denotes the Frobenius semisimplification of the Weil-Deligne representation associated to $r_{l,\iota}(\pi)|_{G_{F_v}}$, as in section 1 of [TY07].

Proof. This follows at once from the main results of [Shi11], [CH13], [Car12a], [BLGGT14a] and [Car12b]. \square

We say that a continuous irreducible representation $r : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_l)$ (respectively $\bar{r} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$) is automorphic if $r \cong r_{l,\iota}(\pi)$ (respectively $\bar{r} \cong \bar{r}_{l,\iota}(\pi)$) for some RACSDC representation π of $\text{GL}_n(\mathbb{A}_F)$. We say that a continuous irreducible representation $r : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_l)$ is automorphic of weight $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \overline{\mathbb{Q}}_l)}$ if $r \cong r_{l,\iota}(\pi)$ for some RACSDC representation π of $\text{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$.

The theory of base change gives a close relationship between automorphic representations of $G(\mathbb{A}_{F^+})$ and automorphic representations of $\text{GL}_n(\mathbb{A}_F)$. For example, one has the following consequences of Corollaire 5.3 and Théorème 5.4 of [Lab09].

Theorem 2.1.3. *Suppose that Π is a RACSDC representation of $\text{GL}_n(\mathbb{A}_F)$ of weight $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \mathbb{C})}$. Then there is an automorphic representation π of $G(\mathbb{A}_{F^+})$ such that*

- (1) For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and each $\tilde{\tau} \hookrightarrow \mathbb{C}$ extending τ , we have $\pi_\tau \cong \Sigma_{\lambda_{\tilde{\tau}}}^\vee \circ \iota_{\tilde{\tau}}$.
- (2) If v is a finite place of F^+ which splits as ww^c in F , then $\pi_v \cong \Pi_w \circ \iota_w$.
- (3) If v is a finite place of F^+ which is inert in F , and Π_v is unramified, then π_v has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$.

Theorem 2.1.4. *Suppose that π is an automorphic representation of $G(\mathbb{A}_{F^+})$. Then either:*

- (1) There is an RACSDC automorphic representation Π of $\text{GL}_n(\mathbb{A}_F)$ of some weight $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \mathbb{C})}$, or:

- (2) *There is a nontrivial partition $n = n_1 + \cdots + n_r$ and cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ such that if $\Pi := \pi_1 \boxplus \cdots \boxplus \pi_r$ is the isobaric direct sum of the π_i , then Π is regular, algebraic, and conjugate self-dual of some weight $\lambda \in (\mathbb{Z}_+^n)_0^{\mathrm{Hom}(F, \mathbb{C})}$*

such that in either case

- (1) *For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and each $\tilde{\tau} \hookrightarrow \mathbb{C}$ extending τ , we have $\pi_\tau \cong \Sigma_{\lambda_{\tilde{\tau}}}^\vee \circ \iota_{\tilde{\tau}}$.*
- (2) *If v is a finite place of F^+ which splits as ww^c in F , then $\pi_v \cong \Pi_w \circ \iota_w$.*
- (3) *If v is a finite place of F^+ which is inert in F , and π_v has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$, then Π_v is unramified.*

We now wish to define what it means for an irreducible representation $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ to be modular of some weight. In order to do so, we return to the spaces of algebraic modular forms considered before. For each place $w|l$ of F , let k_w denote the residue field of F_w . If w lies over a place v of F^+ , write $v = ww^c$. Let $(\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ denote the subset of $(\mathbb{Z}_+^n)^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ consisting of elements a such that for each $w|l$, if $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)$ and $1 \leq i \leq n$ then

$$a_{\sigma, i} + a_{\sigma^c, n+1-i} = 0.$$

We say that an element $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ is a *Serre weight* if for each $w|l$ and each $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)$ we have

$$l-1 \geq a_{\sigma, i} - a_{\sigma, i+1}$$

for all $1 \leq i \leq n-1$. Similarly, if \mathbb{F} is a finite extension of \mathbb{F}_l , we say that an element $a \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(\mathbb{F}, \overline{\mathbb{F}}_l)}$ is a *Serre weight* if for each $\sigma \in \mathrm{Hom}(\mathbb{F}, \overline{\mathbb{F}}_l)$ and each $1 \leq i \leq n-1$ we have

$$l-1 \geq a_{\sigma, i} - a_{\sigma, i+1}.$$

Given any $a \in \mathbb{Z}_+^n$ with $l-1 \geq a_i - a_{i+1}$ for all $1 \leq i \leq n-1$, we define the \mathbb{F} -representation P_a of $\mathrm{GL}_n(\mathbb{F})$ to be the representation obtained by evaluating $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0 a)_{\mathbb{F}}$ on \mathbb{F} , and let N_a be the irreducible sub- \mathbb{F} -representation of P_a generated by the highest weight vector (that this is indeed irreducible follows for example from II.2.8(1) of [Jan03] and the appendix to [Her09]).

If $a \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(\mathbb{F}, \overline{\mathbb{F}}_l)}$ is a Serre weight then we define an irreducible $\overline{\mathbb{F}}_l$ -representation F_a of $\mathrm{GL}_n(\mathbb{F})$ by

$$F_a := \bigotimes_{\tau \in \mathrm{Hom}(\mathbb{F}, \overline{\mathbb{F}}_l)} N_{a_\tau} \otimes_{\mathbb{F}, \tau} \overline{\mathbb{F}}_l.$$

We say that two Serre weights a and b are *equivalent* if and only if $F_a \cong F_b$ as representations of $\mathrm{GL}_n(\mathbb{F})$. This is equivalent to demanding that for each $\sigma \in \mathrm{Hom}(\mathbb{F}, \overline{\mathbb{F}}_l)$, we have

$$a_{\sigma, i} - a_{\sigma, i+1} = b_{\sigma, i} - b_{\sigma, i+1},$$

for each $1 \leq i \leq n-1$, and the character $\mathbb{F}^\times \rightarrow \overline{\mathbb{F}}_l^\times$ given by

$$x \mapsto \prod_{\sigma \in \mathrm{Hom}(\mathbb{F}, \overline{\mathbb{F}}_l)} \sigma(x)^{a_{\sigma, n} - b_{\sigma, n}}$$

is trivial. Every irreducible $\overline{\mathbb{F}}_l$ -representation of $\mathrm{GL}_n(\mathbb{F})$ is of the form F_a for some a (see for example the appendix to [Her09]).

If $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \text{Hom}(k_w, \overline{\mathbb{F}}_l)}$ is a Serre weight, we define an irreducible $\overline{\mathbb{F}}_l$ -representation F_a of $G(\mathcal{O}_{F^+, l})$ as follows: we define

$$F_a = \otimes_{\overline{\mathbb{F}}_l} F_{a_{\tilde{v}}},$$

an irreducible representation of $\prod_{\tilde{v} \in \tilde{S}_l} \text{GL}_n(k_v)$, and we let $G(\mathcal{O}_{F^+, l})$ act on $F_{a_{\tilde{v}}}$ by the composition of $\iota_{\tilde{v}}$ and reduction modulo l . Again, we say that two Serre weights a and b are equivalent if and only if $F_a \cong F_b$ as representations of $G(\mathcal{O}_{F^+, l})$. This is equivalent to demanding that for each place $w|l$ and each $\sigma \in \text{Hom}(k_w, \overline{\mathbb{F}}_l)$ and each $1 \leq i \leq n-1$ we have

$$a_{\sigma, i} - a_{\sigma, i+1} = b_{\sigma, i} - b_{\sigma, i+1},$$

and the character $k_w^\times \rightarrow \overline{\mathbb{F}}_l^\times$ given by

$$x \mapsto \prod_{\sigma \in \text{Hom}(k_w, \overline{\mathbb{F}}_l)} \sigma(x)^{a_{\sigma, n} - b_{\sigma, n}}$$

is trivial.

Note that the representation F_a is independent of the choice of \tilde{S}_l (this follows easily from the condition that $a_{\sigma c, n+1-i} = -a_{\sigma, i}$ and the relation $\iota_{w^c}(x) = {}^t(\iota_w(x))^{-1}$).

For future use, if $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \text{Hom}(k_w, \overline{\mathbb{F}}_l)}$ is a Serre weight, we also define an $\overline{\mathbb{F}}_l$ -representation P_a of $G(\mathcal{O}_{F^+, l})$ as follows: we define

$$P_a = \otimes_{\overline{\mathbb{F}}_l} P_{a_{\tilde{v}}},$$

a representation of $\prod_{\tilde{v} \in \tilde{S}_l} \text{GL}_n(k_v)$, and we let $G(\mathcal{O}_{F^+, l})$ act on $P_{a_{\tilde{v}}}$ by the composition of $\iota_{\tilde{v}}$ and reduction modulo l . Note that F_a is a subrepresentation of P_a .

We say that a weight $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \overline{\mathbb{Q}}_l)}$ is a *lift* of a Serre weight a if for each $w|l$ and each $\sigma \in \text{Hom}(k_w, \overline{\mathbb{F}}_l)$ there is an element $\tau \in \text{Hom}(F, \overline{\mathbb{Q}}_l)$ lying over w and lifting σ such that $\lambda_\tau = a_\sigma$, and for all other $\tau' \in \text{Hom}(F, \overline{\mathbb{Q}}_l)$ lying over w and lifting σ we have $\lambda_{\tau'} = 0$. If $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \overline{\mathbb{Q}}_l)}$ and $w|l$ is a place of F , we let $\lambda_w \in (\mathbb{Z}_+^n)^{\text{Hom}(F_w, \overline{\mathbb{Q}}_l)}$ be defined in the obvious way. If L is a finite extension of \mathbb{Q}_l with residue field k_L , we say that an element $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(L, \overline{\mathbb{Q}}_l)}$ is a *lift* of an element $a \in (\mathbb{Z}_+^n)^{\text{Hom}(k_L, \overline{\mathbb{F}}_l)}$ if for each $\sigma \in \text{Hom}(k_L, \overline{\mathbb{F}}_l)$ there is an element $\tau \in \text{Hom}(L, \overline{\mathbb{Q}}_l)$ lifting σ such that $\lambda_\tau = a_\sigma$, and for all other $\tau' \in \text{Hom}(L, \overline{\mathbb{Q}}_l)$ lifting σ we have $\lambda_{\tau'} = 0$.

For the rest of this section, fix $K = \overline{\mathbb{Q}}_l$.

Definition 2.1.5. We say that a compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$ is *good* if $U = \prod_v U_v$ with U_v a compact open subgroup of $G(F_v^+)$ such that:

- $U_v \subset G(\mathcal{O}_{F_v^+})$ for all v which split in F ;
- $U_v = G(\mathcal{O}_{F_v^+})$ if $v|l$;
- U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$ if v is inert in F .

Let U be a good compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$. Let T be a finite set of finite places of F^+ which split in F , containing S_l and all the places v which split in F for which $U_v \neq G(\mathcal{O}_{F_v^+})$. We let $\mathbb{T}^{T, \text{univ}}$ be the commutative \mathcal{O} -polynomial algebra generated by formal variables $T_w^{(j)}$ for all $1 \leq j \leq n$, w a place of F lying over a

place v of F^+ which splits in F and is not contained in T . For any $\lambda \in (\mathbb{Z}_+^n)^{\tilde{l}}$, the algebra $\mathbb{T}^{T, \text{univ}}$ acts on $S_\lambda(U, \mathcal{O})$ via the Hecke operators

$$T_w^{(j)} := \iota_w^{-1} \left[\text{GL}_n(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \text{GL}_n(\mathcal{O}_{F_w}) \right]$$

for $w \notin T$ and ϖ_w a uniformiser in \mathcal{O}_{F_w} . Similarly, for any Serre weight $a \in (\mathbb{Z}_+^n)_{\prod_{v|l} \text{Hom}(k_v, \mathbb{F}_l)}$, $\mathbb{T}^{T, \text{univ}}$ acts on $S(U, F_a)$.

Suppose that \mathfrak{m} is a maximal ideal of $\mathbb{T}^{T, \text{univ}}$ with residue field $\overline{\mathbb{F}}_l$ such that $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$. Then (cf. Proposition 3.4.2 of [CHT08]) by Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a continuous semisimple representation

$$\bar{r}_{\mathfrak{m}} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$$

associated to \mathfrak{m} , which is uniquely determined by the properties that:

- $\bar{r}_{\mathfrak{m}}^c \cong \bar{r}_{\mathfrak{m}}^\vee \epsilon_l^{1-n}$,
- for all finite places w of F not lying over T , $\bar{r}_{\mathfrak{m}}|_{G_{F_w}}$ is unramified, and
- if w is a finite place of F which doesn't lie over T and which splits over F^+ , then the characteristic polynomial of $\bar{r}_{\mathfrak{m}}(\text{Frob}_w)$ is

$$X^n - T_w^{(1)} X^{n-1} + \dots + (-1)^j (\mathbf{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \dots + (-1)^n (\mathbf{N}w)^{n(n-1)/2} T_w^{(n)}.$$

Lemma 2.1.6. *Suppose that U is sufficiently small, and let \mathfrak{m} be a maximal ideal of $\mathbb{T}_\lambda^{T, \text{univ}}$ with residue field $\overline{\mathbb{F}}_l$. Suppose that $a \in (\mathbb{Z}_+^n)_{\prod_{v|l} \text{Hom}(k_v, \mathbb{F}_l)}$ is a Serre weight, and that $\lambda \in (\mathbb{Z}_+^n)^{\tilde{l}}$ is a lift of a . Then*

$$S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$$

if and only if for some Jordan-Hölder factor F of the $G(\mathcal{O}_{F^+, l})$ -representation P_a ,

$$S(U, F)_{\mathfrak{m}} \neq 0.$$

In particular if $S(U, F_a)_{\mathfrak{m}} \neq 0$ then $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$.

Proof. We have $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} = S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \otimes \overline{\mathbb{Q}}_l$. Since U is sufficiently small, it follows that $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}}$ is l -torsion free. Thus $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$ if and only if $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \neq 0$. However, using the fact that U is sufficiently small again, we have $S_\lambda(U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} \neq 0$ if and only if $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \neq 0$. Thus, $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$ if and only if $S_\lambda(U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} \neq 0$.

But $S_\lambda(U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} = S(U, W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ is nonzero if and only if $S(U, F)_{\mathfrak{m}}$ is nonzero for some Jordan-Hölder factor F of $W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$. (This follows from the exactness of the functor $F \mapsto S(U, F)_{\mathfrak{m}}$ which in turn follows from the fact that U is sufficiently small.) It then suffices to note that as an immediate consequence of the definitions, we have $P_a \cong W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$ and F_a is a Jordan-Hölder factor of $W_\lambda \otimes_{\mathcal{O}} \mathbb{F}$. \square

We have the following definitions.

Definition 2.1.7. If R is a commutative ring and $r : G_F \rightarrow \text{GL}_n(R)$ is a representation, we say that r has *split ramification* if $r|_{G_{F_w}}$ is unramified for any finite place $w \in F$ which does not split over F^+ .

Definition 2.1.8. If π is a RACSDC automorphic representation of $\text{GL}_n(\mathbb{A}_F)$, we say that π has *split ramification* if π_w is unramified for any finite place $w \in F$ which does not split over F^+ .

Definition 2.1.9. Suppose that $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_l)$ is a continuous irreducible representation. Then we say that \bar{r} is *modular of weight* $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \bar{\mathbb{F}}_l)}$ if there is a good, sufficiently small level U , a set of places T as above, and a maximal ideal \mathfrak{m} of $\mathbb{T}^{T, \mathrm{univ}}$ with residue field $\bar{\mathbb{F}}_l$ such that

- $S(U, F_a)_{\mathfrak{m}} \neq 0$, and
- $\bar{r} \cong \bar{r}_{\mathfrak{m}}$.

(Note that $\bar{r}_{\mathfrak{m}}$ exists by Lemma 2.1.6 and the remarks preceding it.) We say that \bar{r} is modular if it is modular of some weight.

Remark 2.1.10. Note that if $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_l)$ is modular then \bar{r} must have split ramification, and $\bar{r}^c \cong \bar{r}^{\vee} \bar{\epsilon}_l^{1-n}$. Note also that this definition is independent of the choice of \tilde{S}_l (because F_a is independent of this choice).

Lemma 2.1.11. *Suppose that $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_l)$ is a continuous irreducible representation with split ramification. Let $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \bar{\mathbb{F}}_l)}$ be a Serre weight, and let $\lambda \in (\mathbb{Z}_+^n)_0^{\mathrm{Hom}(F, \bar{\mathbb{Q}}_l)}$ be a lift of a . Then if \bar{r} is modular of weight $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \bar{\mathbb{F}}_l)}$, there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$ and level prime to l which has split ramification, and which satisfies $\bar{r}_{l, \iota}(\pi) \cong \bar{r}$. Conversely, if there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$ and level prime to l which has split ramification, and which satisfies $\bar{r}_{l, \iota}(\pi) \cong \bar{r}$, then \bar{r} is modular of weight $b \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \bar{\mathbb{F}}_l)}$ for some b such that the $G(\mathcal{O}_{F^+, l})$ -representation P_a has a Jordan-Hölder factor isomorphic to F_b .*

Proof. Suppose firstly that \bar{r} is modular of weight a . Then by definition there is a good U and a T as above with U sufficiently small, and a maximal ideal \mathfrak{m} of $\mathbb{T}^{T, \mathrm{univ}}$ with residue field $\bar{\mathbb{F}}_l$ such that

- $S(U, F_a)_{\mathfrak{m}} \neq 0$, and
- $\bar{r} \cong \bar{r}_{\mathfrak{m}}$.

By Lemma 2.1.6, the first property implies that $S_{\lambda}(U, \bar{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$. Define a compact open subgroup $U' = \prod_w U'_w$ of $\mathrm{GL}_n(\mathbb{A}_F^{\infty})$ by

- $U'_w = \mathrm{GL}_n(\mathcal{O}_{F_w})$ if w is not split over F^+ .
- $U'_w = \iota_w(U_w)$ if w splits over F^+ .

By Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a RACSDC automorphic representation π of weight λ which satisfies $\bar{r}_{l, \iota}(\pi) \cong \bar{r}$, and $\pi_w^{U'_w} \neq 0$ for all finite places w of F . Since U is good, we see that π has level prime to l , and it has split ramification, as required.

Conversely, suppose that there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight λ which has split ramification and level prime to l with $\bar{r}_{l, \iota}(\pi) \cong \bar{r}$. Then there is a compact open subgroup $U' = \prod_w U'_w$ of $\mathrm{GL}_n(\mathbb{A}_F^{\infty})$ such that

- for each finite place w of F , $\pi_w^{U'_w} \neq 0$,
- $U'_w \subset \mathrm{GL}_n(\mathcal{O}_{F_w})$ for all w ,
- $U'_w = \mathrm{GL}_n(\mathcal{O}_{F_w})$ for all $w|l$ and all w which are not split over F^+ ,
- if $v = ww^c$ is a place of F^+ which splits in F , then $U_{w^c} = c({}^t U_w^{-1})$,
- there is a finite place w of F which is split over F^+ such that
 - w lies above a rational prime p with $[F(\zeta_p) : F] > n$, and

$$- U'_w = \ker(\mathrm{GL}_n(\mathcal{O}_w) \rightarrow \mathrm{GL}_n(\mathcal{O}_w/\varpi_w)).$$

Define a compact open subgroup $U = \prod_v U_v$ of $G(\mathbb{A}_{F^+}^\infty)$ by

- if v is inert in F , then $U_v = G(\mathcal{O}_{F_v^+})$, and
- if $v = ww^c$ splits in F , then $U_v = \iota_w^{-1}(U'_w)$ (which is well-defined by the fourth bullet point above).

By the final bullet point in the list of properties of U' above, U is sufficiently small. Then by Lemma 2.1.1 and Theorem 2.1.3 we have $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$. The result now follows from Lemma 2.1.6. \square

3. A LIFTING THEOREM

3.1. We recall some terminology from [BLGGT14b], specialized to the crystalline (as opposed to potentially crystalline) case. Fix a prime l . Let K be a finite extension of \mathbb{Q}_l , and \mathcal{O} the ring of integers in a finite extension of \mathbb{Q}_l inside $\overline{\mathbb{Q}}_l$, with residue field k . Assume that for each continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_l$, the image is contained in the field of fractions of \mathcal{O} .

Let $\bar{\rho} : G_K \rightarrow \mathrm{GL}_n(k)$ be a continuous representation, and let $R_{\mathcal{O}, \bar{\rho}}^\square$ be the universal \mathcal{O} -lifting ring. Let $\{H_\tau\}$ be a collection of n element multisets of integers parametrized by $\tau \in \mathrm{Hom}_{\mathbb{Q}_l}(K, \overline{\mathbb{Q}}_l)$. Then $R_{\mathcal{O}, \bar{\rho}}^\square$ has a unique quotient $R_{\mathcal{O}, \bar{\rho}, \{H_\tau\}, \mathrm{cris}}^\square$ which is reduced and without l -torsion and such that a $\overline{\mathbb{Q}}_l$ -point of $R_{\mathcal{O}, \bar{\rho}}^\square$ factors through $R_{\mathcal{O}, \bar{\rho}, \{H_\tau\}, \mathrm{cris}}^\square$ if and only if it corresponds to a representation $\rho : G_K \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ which is crystalline and has $\mathrm{HT}_\tau(\rho) = H_\tau$ for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$. We will write $R_{\bar{\rho}, \{H_\tau\}, \mathrm{cris}}^\square \otimes \overline{\mathbb{Q}}_l$ for $R_{\mathcal{O}, \bar{\rho}, \{H_\tau\}, \mathrm{cris}}^\square \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$. This definition is independent of the choice of \mathcal{O} . The scheme $\mathrm{Spec}(R_{\bar{\rho}, \{H_\tau\}, \mathrm{cris}}^\square \otimes \overline{\mathbb{Q}}_l)$ is formally smooth over $\mathrm{Spec} \overline{\mathbb{Q}}_l$. (See [Kis08].)

Let $\rho_1, \rho_2 : G_K \rightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ be continuous representations. We say that ρ_1 connects to ρ_2 , which we denote $\rho_1 \sim \rho_2$, if and only if

- the reduction $\bar{\rho}_1 = \rho_1 \bmod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_l}}$ is equivalent to the reduction $\bar{\rho}_2 = \rho_2 \bmod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_l}}$;
- ρ_1 and ρ_2 are both crystalline;
- for each $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\rho_1) = \mathrm{HT}_\tau(\rho_2)$;
- and ρ_1 and ρ_2 define points on the same irreducible component of the scheme $\mathrm{Spec}(R_{\bar{\rho}_1, \{\mathrm{HT}_\tau(\rho_1)\}, \mathrm{cris}}^\square \otimes \overline{\mathbb{Q}}_l)$.

We note that $\rho_1 \sim \rho_2$ in our sense if and only if both ρ_1 and ρ_2 are crystalline and $\rho_1 \sim \rho_2$ in the sense of [BLGGT14b]. As in section 2.3 of [BLGGT14b], we have the following:

- (1) The relation $\rho_1 \sim \rho_2$ does not depend on the equivalence chosen between the reductions $\bar{\rho}_1$ and $\bar{\rho}_2$, nor on the $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ -conjugacy class of ρ_1 or ρ_2 .
- (2) \sim is symmetric and transitive.
- (3) If K'/K is a finite extension and $\rho_1 \sim \rho_2$ then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$.
- (4) If $\rho_1 \sim \rho_2$ and $\rho'_1 \sim \rho'_2$ then $\rho_1 \oplus \rho'_1 \sim \rho_2 \oplus \rho'_2$ and $\rho_1 \otimes \rho'_1 \sim \rho_2 \otimes \rho'_2$ and $\rho_1^\vee \sim \rho_2^\vee$.
- (5) If $\mu : G_K \rightarrow \overline{\mathbb{Q}}_l^\times$ is a continuous unramified character with $\bar{\mu} = 1$ then $\rho_1 \sim \rho_1 \otimes \mu$.
- (6) Suppose ρ_1 is crystalline and $\bar{\rho}_1$ is semisimple. Let Fil^i be a G_K -invariant filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$ -direct summands. Then $\rho_1 \sim \oplus_i \mathrm{gr}^i(\mathrm{Fil})$.

We will call a crystalline representation $\rho : G_K \rightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonal* if it is of the form $\chi_1 \oplus \cdots \oplus \chi_n$ with $\chi_i : G_K \rightarrow \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$. We will call a crystalline representation $\rho : G_K \rightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonalizable* if it connects to some diagonal representation. We will call a representation $\rho_1 : G_K \rightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *potentially diagonalizable* if there is a finite extension K'/K such that $\rho_1|_{G_{K'}}$ is diagonalizable. Note that if K''/K is a finite extension and ρ_1 is diagonalizable (resp. potentially diagonalizable) then $\rho_1|_{G_{K''}}$ is diagonalizable (resp. potentially diagonalizable).

Suppose now that K is a finite extension of \mathbb{Q}_p for some prime $p \neq l$ and

$$\rho_1, \rho_2 : G_K \rightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

are two continuous representations. We define the notion that ρ_1 connects to ρ_2 exactly as in [BLGGT14b]. Again, this will be denoted by $\rho_1 \sim \rho_2$.

Recall the following definition from [Tho12] (for a discussion of the equivalence of this definition to that formulated in [Tho12], see the appendix to [BLGG13]).

Definition 3.1.1. We call a finite subgroup $H \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ *adequate* if the following conditions are satisfied.

- (1) H has no non-trivial quotient of l -power order (i.e. $H^1(H, \overline{\mathbb{F}}_l) = (0)$).
- (2) $l \nmid n$.
- (3) The elements of H with order coprime to l span $M_{n \times n}(\overline{\mathbb{F}}_l)$ over $\overline{\mathbb{F}}_l$. (This implies that $\overline{\mathbb{F}}_l^n$ is an irreducible representation of H .)
- (4) $H^1(H, \mathfrak{gl}_n(\overline{\mathbb{F}}_l)) = (0)$.

In particular, we have the following useful result, an immediate consequence of Theorem 9 of [GHTT10].

Theorem 3.1.2. *Suppose that $l \geq 2(n+1)$, and that H is a finite subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ which acts irreducibly. Then H is adequate.*

Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \rightarrow \mathbb{C}$. Let F be an imaginary CM field with maximal totally real subfield F^+ .

Theorem 3.1.3. *Let $l > 2$ be prime, and let F be a CM field with maximal totally real subfield F^+ , with $\zeta_l \notin F$. Assume that the extension F/F^+ is split at all places dividing l . Suppose that*

$$\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation which satisfies the following properties.

- (1) *There is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
 - $\bar{r} \cong \bar{r}_{l,\iota}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$).
 - *For each place $w|l$ of F , $r_{l,\iota}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable.*
- (2) *The image $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let S be a finite set of finite places of F^+ which split in F . Assume that S contains all the places of F^+ dividing l , and all places lying under a place of F at which \bar{r} is ramified. For each $v \in S$ choose a place \tilde{v} of F above v , and a lift $\rho_{\tilde{v}} : G_{F_{\tilde{v}}} \rightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of $\bar{r}|_{G_{F_{\tilde{v}}}}$. Assume that if $v|l$, then $\rho_{\tilde{v}}$ is crystalline and potentially diagonalizable, and if $\tau : F_{\tilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$ is any embedding, then $\mathrm{HT}_\tau(\rho_{\tilde{v}})$ consists of distinct integers.

Then there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of level prime to l such that

- $\bar{r} \cong \bar{r}_{l,\iota}(\pi)$.
- π_w is unramified for all w not lying over a place of S , so that $r_{l,\iota}(\pi_w)$ is unramified at all such w .
- $r_{l,\iota}(\pi)|_{G_{F_{\bar{v}}}} \sim \rho_{\bar{v}}$ for all $v \in S$. In particular, for each place $v|l$, $r_{l,\iota}(\pi)|_{G_{F_{\bar{v}}}}$ is crystalline and for each embedding $\tau : F_{\bar{v}} \hookrightarrow \overline{\mathbb{Q}_l}$, $\text{HT}_{\tau}(r_{l,\iota}(\pi)|_{G_{F_{\bar{v}}}}) = \text{HT}_{\tau}(\rho_{\bar{v}})$.

Proof. Let \mathcal{G}_n be the group scheme over \mathbb{Z} defined in section 2.1 of [CHT08]. Then by the main result of [BC11], \bar{r} extends to a representation $\bar{\rho} : G_{F^+} \rightarrow \mathcal{G}_n(\overline{\mathbb{F}_l})$ with multiplier $\bar{\epsilon}_l^{1-n}$.

We now apply Theorem A.4.1 of [BLGG13], with

- F , n and S as in the present setting.
- \bar{r} our present $\bar{\rho}$.
- ρ_v our $\rho_{\bar{v}}$.
- $\mu = \epsilon_l^{1-n}$.
- $F' = F$.

We conclude that \bar{r} has a lift $r : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}_l})$ (the restriction to G_F of the representation r of Theorem A.4.1 of [BLGG13]) such that

- $r^c \cong r^{\vee} \epsilon_l^{1-n}$.
- if $v \in S$ then $r|_{G_{F_{\bar{v}}}} \sim \rho_{\bar{v}}$.
- r is unramified outside S .
- r is automorphic of level potentially prime to l .

By Theorem 2.1.2, we see that (since $r|_{G_{F_w}}$ is crystalline for all $w|l$, and unramified at all places w not lying over a place in S) π_w is unramified for all $w|l$ and all w not lying over a place in S , as required. \square

4. SERRE WEIGHT CONJECTURES

4.1. We now briefly discuss Serre weight conjectures for GL_n . We refer the reader to the forthcoming [EGHS14] for a far more detailed discussion. In particular, in much of this section we restrict ourselves to the case that l splits completely in F , both for simplicity of notation and because in this case we can prove theorems with cleaner conditions, as representations satisfying the Fontaine-Laffaille condition are always potentially diagonalizable.

Let K be a finite extension of \mathbb{Q}_l , with ring of integers \mathcal{O}_K and residue field k . Let $\bar{\rho} : G_K \rightarrow \text{GL}_n(\overline{\mathbb{F}_l})$ be a continuous representation. Then it is a folklore conjecture that for each such $\bar{\rho}$, there is a set $W(\bar{\rho})$ of Serre weights of $\text{GL}_n(k)$ for each K and each $\bar{\rho}$ with the following property: if F is a CM field, $\bar{r} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}_l})$ is an irreducible modular representation (so in particular it is conjugate self-dual), $w|l$ is a place of F and σ_w is an irreducible $\overline{\mathbb{F}_l}$ -representation of $\text{GL}_n(k_w)$, then \bar{r} is modular of Serre weight $\sigma_w \otimes_{\overline{\mathbb{F}_l}} \sigma^w$ for some σ^w if and only if $\sigma_w \in W(\bar{r}|_{G_{F_w}})$.

It is natural to believe that there is a description of $W(\bar{\rho})$ in terms of the existence of crystalline lifts with particular Hodge-Tate weights, as we now explain. This is one of the motivations for the general Serre weight conjectures explained in [EGHS14].

Definition 4.1.1. Let K/\mathbb{Q}_l be a finite extension, let $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(K, \overline{\mathbb{Q}_l})}$, and let $\rho : G_K \rightarrow \text{GL}_n(\overline{\mathbb{Q}_l})$ be a de Rham representation. Then we say that ρ has *Hodge*

type λ if for each $\tau \in \text{Hom}(K, \overline{\mathbb{Q}}_l)$, we have $\text{HT}_\tau(\rho) = \{\lambda_{\tau,1} + (n-1), \lambda_{\tau,2} + (n-2), \dots, \lambda_{\tau,n}\}$.

Remark 4.1.2. As an immediate consequence of this definition and of Theorem 2.1.2, we see that if π is a RACSDC automorphic representation of weight $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \mathbb{C})}$, then for each place $w|l$, $r_{l,\iota}(\pi)|_{G_{F_w}}$ has Hodge type $(\iota^{-1}\lambda)_w$.

Lemma 4.1.3. *Let n be a positive integer, and let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F , and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Suppose that $\bar{r} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \text{Hom}(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight, and let $\lambda \in (\mathbb{Z}_+^n)_0^{\text{Hom}(F, \overline{\mathbb{Q}}_l)}$ be a lift of a . If \bar{r} is modular of weight a , then for each place $w|l$ there is a continuous lift $r_w : G_{F_w} \rightarrow \text{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of $\bar{r}|_{G_{F_w}}$ such that r_w is crystalline of Hodge type λ_w .*

Proof. By Lemma 2.1.11 there is a RACSDC automorphic representation π of $\text{GL}_n(\mathbb{A}_F)$, which has level prime to l and weight $\iota\lambda$, such that $\bar{r}_{l,\iota}(\pi) \cong \bar{r}$. Then we may take $r_w := r_{l,\iota}(\pi)|_{G_{F_w}}$, which has the required properties by Remark 4.1.2. \square

This suggests the following definition.

Definition 4.1.4. Let K be a finite extension of \mathbb{Q}_l , with ring of integers \mathcal{O}_K and residue field k . Let $\bar{\rho} : G_K \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation. Then we let $W^{\text{cris}}(\bar{\rho})$ be the set of Serre weights $a \in (\mathbb{Z}_+^n)^{\text{Hom}(k, \overline{\mathbb{F}}_l)}$ with the property that there is a crystalline representation $\rho : G_K \rightarrow \text{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ lifting $\bar{\rho}$, such that ρ has Hodge type λ for some lift $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(K, \overline{\mathbb{Q}}_l)}$ of a .

The results of section 3 suggest the following definition.

Definition 4.1.5. Let K be a finite extension of \mathbb{Q}_l , with ring of integers \mathcal{O}_K and residue field k . Let $\bar{\rho} : G_K \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation. Then we let $W^{\text{diag}}(\bar{\rho})$ be the set of Serre weights $a \in (\mathbb{Z}_+^n)^{\text{Hom}(k, \overline{\mathbb{F}}_l)}$ with the property that there is a potentially diagonalizable crystalline representation $\rho : G_K \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_l)$ lifting $\bar{\rho}$, such that ρ has Hodge type λ for some lift $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(K, \overline{\mathbb{Q}}_l)}$ of a .

Remark 4.1.6. If a and b are equivalent Serre weights, then $a \in W^{\text{cris}}(\bar{\rho})$ (respectively $W^{\text{diag}}(\bar{\rho})$) if and only if $b \in W^{\text{cris}}(\bar{\rho})$ (respectively $W^{\text{diag}}(\bar{\rho})$). This is an easy consequence of Lemma 4.1.15 of [BLGG13], which provides a crystalline character with trivial reduction with which one can twist the crystalline Galois representations of Hodge type some lift of a to obtain crystalline representations of Hodge type some lift of b . The same remarks apply to the set $W^{\text{explicit}}(\bar{\rho})$ defined below.

By definition we have $W^{\text{diag}}(\bar{\rho}) \subset W^{\text{cris}}(\bar{\rho})$. We “globalise” these definitions in the obvious way:

Definition 4.1.7. Let $\bar{r} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation with $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$. Then we let $W^{\text{cris}}(\bar{r})$ (respectively $W^{\text{diag}}(\bar{r})$) be the set of Serre weights $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \text{Hom}(k_w, \overline{\mathbb{F}}_l)}$ such that for each place $w|l$, the corresponding Serre weight $a_w \in (\mathbb{Z}_+^n)^{\text{Hom}(k_w, \overline{\mathbb{F}}_l)}$ is an element of $W^{\text{cris}}(\bar{r}|_{G_{F_w}})$ (respectively $W^{\text{diag}}(\bar{r}|_{G_{F_w}})$).

The point of these definitions is the following Corollary and Theorem.

Corollary 4.1.8. *Let n be a positive integer, let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F , and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Suppose that $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. If \bar{r} is modular of weight a , then $a \in W^{\mathrm{cris}}(\bar{r})$.*

Proof. This is an immediate consequence of Lemma 4.1.3 and Definition 4.1.7. \square

Theorem 4.1.9. *Let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F , and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Assume that $\zeta_l \notin F$. Suppose that $l > 2$, and that $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible representation with split ramification. Assume that*

- *There is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
 - *$\bar{r} \cong \bar{r}_{l,\iota}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$).*
 - *For each place $w|l$ of F , $r_{l,\iota}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable.*
 - *$\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. Assume that $a \in W^{\mathrm{diag}}(\bar{r})$. Then there is a Serre weight $b \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ such that

- *\bar{r} is modular of weight b .*
- *There is a Jordan-Hölder factor of the $G(\mathcal{O}_{F^+,l})$ representation P_a which is isomorphic to F_b .*

Proof. By the assumption that $a \in W^{\mathrm{diag}}(\bar{r})$, there is a lift λ of a such that for each $w|l$ there is a potentially diagonalizable crystalline lift $\rho_w : G_{F_w} \rightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of $\bar{r}|_{G_{F_w}}$ of Hodge type λ_w .

By Theorem 3.1.3, there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$, of level prime to l and with split ramification, such that $\bar{r}(\pi) \cong \bar{r}$. The result follows from Lemma 2.1.11. \square

Since Fontaine–Laffaille representations are potentially diagonalizable, we obtain the following Corollary.

Corollary 4.1.10. *Let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F , and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Suppose that $l > 2$, and that $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible representation with split ramification. Assume that*

- (1) *l is unramified in F .*
- (2) *There is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\mu \in (\mathbb{Z}_+^n)_0^{\mathrm{Hom}(F, \mathbb{C})}$ and level prime to l such that*
 - *$\bar{r} \cong \bar{r}_{l,\iota}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$).*
 - *For each $\tau \in \mathrm{Hom}(F, \mathbb{C})$, $\mu_{\tau,1} - \mu_{\tau,n} \leq l - n$.*
 - *$\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. Assume that $a \in W^{\mathrm{diag}}(\bar{r})$. Then there is a Serre weight $b \in (\mathbb{Z}_+^n)_0^{\prod_{w|l} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_l)}$ such that

- \bar{r} is modular of weight b .
- There is a Jordan-Hölder factor of the $G(\mathcal{O}_{F^+,l})$ representation P_a which is isomorphic to F_b .

Proof. By Theorem 4.1.9, it is enough to check that for each place $w|l$ of F , $r_{l,l}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable. This follows from the main result of [GL12]. \square

As explained above, we now specialise to the case that l splits completely in F . We further assume that $\bar{r}|_{G_{F_w}}$ is semisimple for all $w|l$, and specify a set $W^{\text{explicit}}(\bar{r})$ of Serre weights. These weights will have the property that if $a \in W^{\text{explicit}}(\bar{r})$, and λ is the unique lift of a to $(\mathbb{Z}_+^n)^{\text{Hom}(F, \overline{\mathbb{Q}}_l)}$, then for each place $w|l$, $\bar{r}|_{G_{F_w}}$ has a potentially diagonalizable (indeed potentially diagonal) crystalline lift of Hodge type λ_w .

Since the situation is purely local, we change notation and work with $G_{\mathbb{Q}_l}$. Let \mathbb{Q}_{l^m} denote the unramified extension of \mathbb{Q}_l of degree m inside $\overline{\mathbb{Q}}_l$, and let $\omega_m : G_{\mathbb{Q}_{l^m}} \rightarrow \overline{\mathbb{F}}_l^\times$ denote a choice of fundamental character of niveau m (this is given by the action of $G_{\mathbb{Q}_{l^m}}$ on the $(l^m - 1)$ -st roots of l). Given $\lambda \in \overline{\mathbb{F}}_l^\times$ and an m -tuple of integers $\underline{c} = (c_0, \dots, c_{m-1})$, we consider the representation

$$\bar{\rho}_{\lambda, \underline{c}} := \text{nr}_\lambda \otimes \text{Ind}_{G_{\mathbb{Q}_{l^m}}}^{G_{\mathbb{Q}_l}} \omega_m^{-(c_0 + lc_1 + \dots + l^{m-1}c_{m-1})},$$

where nr_λ is the unramified character taking a geometric Frobenius to λ . Given a partition $\underline{n} = n_1 + \dots + n_r$, elements $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ of $\overline{\mathbb{F}}_l^\times$, and a tuple $\underline{c} = (\underline{c}_1, \dots, \underline{c}_r)$ of tuples $\underline{c}_i = (c_{i,0}, \dots, c_{i,n_i-1})$ of integers, we define the representation

$$\rho_{\underline{n}, \underline{\lambda}, \underline{c}} := \bigoplus_{i=1}^r \rho_{\lambda_i, \underline{c}_i}.$$

Note that we can think of \underline{c} as the element $(c_{1,0}, c_{1,2}, \dots, c_{r,n_r-1})$ of \mathbb{Z}^n , where $n = n_1 + \dots + n_r$.

Definition 4.1.11. Let $\bar{\rho} : G_{\mathbb{Q}_l} \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$ be a semisimple representation. Let $W^{\text{explicit}}(\bar{\rho})$ be the set of Serre weights $a \in \mathbb{Z}_+^n$ for which there exists a permutation $\sigma \in S_n$, a partition \underline{n} of n and $\underline{\lambda}$ as above such that

$$\bar{\rho} \cong \bar{\rho}_{\underline{n}, \underline{\lambda}, (a_{\sigma(1)} + n - \sigma(1), \dots, a_{\sigma(n)} + n - \sigma(n))}.$$

We let $W_I^{\text{explicit}}(\bar{\rho})$ be the set of Serre weights a for which there exist σ , \underline{n} and $\underline{\lambda}$ such that

$$\bar{\rho}|_{I_{\mathbb{Q}_l}} \cong \bar{\rho}_{\underline{n}, \underline{\lambda}, (a_{\sigma(1)} + n - \sigma(1), \dots, a_{\sigma(n)} + n - \sigma(n))}|_{I_{\mathbb{Q}_l}};$$

this notion will be useful in Section 4.2.

Lemma 4.1.12. *If $\bar{\rho} : G_{\mathbb{Q}_l} \rightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$ is a semisimple representation and $a \in W^{\text{explicit}}(\bar{\rho})$, then $\bar{\rho}$ has a potentially diagonalizable crystalline lift of Hodge type a .*

Proof. By the definition of ‘‘Hodge type a ’’, it is enough to show that each representation $\bar{\rho}_{\lambda, \underline{c}} : G_{\mathbb{Q}_l} \rightarrow \text{GL}_m(\overline{\mathbb{F}}_l)$ defined above has a potentially diagonalizable crystalline lift with Hodge–Tate weights c_0, \dots, c_{m-1} (note that the direct sum of potentially diagonalizable representations is again potentially diagonalizable). It thus suffices to show that the character $\omega_m^{-(c_0 + lc_1 + \dots + l^{m-1}c_{m-1})}$ of $G_{\mathbb{Q}_{l^m}}$ has a crystalline lift with Hodge–Tate weights c_0, \dots, c_{m-1} (because the induction to $G_{\mathbb{Q}_l}$ of such a lift is certainly potentially diagonalizable). This follows at once from Lemma

6.2 of [GS11] (noting that the conventions on the sign of Hodge–Tate weights in [GS11] are the opposite of those of this paper). \square

Again we may globalise this definition in the obvious way.

Definition 4.1.13. Continue to assume that l splits completely in F , and let $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_l)$ be a continuous representation with $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$ and such that $\bar{r}|_{G_{F_w}}$ is semisimple for each $w|l$. Then we let $W^{\mathrm{explicit}}(\bar{r})$ be the set of Serre weights $a \in (\mathbb{Z}_+^n)_0^{\coprod_{w|l} \mathrm{Hom}(k_w, \bar{\mathbb{F}}_l)}$ such that for each place $w|l$, the corresponding Serre weight $a_w \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(k_w, \bar{\mathbb{F}}_l)}$ is an element of $W^{\mathrm{explicit}}(\bar{r}|_{G_{F_w}})$.

Corollary 4.1.14. Let $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_l)$ be a continuous representation satisfying the assumptions of Definition 4.1.13. Then $W^{\mathrm{explicit}}(\bar{r}) \subset W^{\mathrm{diag}}(\bar{r})$.

Proof. This follows immediately from Lemma 4.1.12. \square

In the case $n = 2$, which we explored more thoroughly in [BLGG13], $W^{\mathrm{explicit}}(\bar{r})$ is precisely the set of weights for which \bar{r} is modular. We do not conjecture this for $n > 2$; even for $n = 3$ one sees that the set of weights predicted in [Her09] is larger than $W^{\mathrm{explicit}}(\bar{r})$. In fact, we expect (see [EGHS14] for a much more detailed discussion) that the set of weights for which \bar{r} is modular is $W^{\mathrm{cris}}(\bar{r})$, and it is easy to see that this set is typically larger than $W^{\mathrm{explicit}}(\bar{r})$. Indeed, by Lemma 2.1.11 and Theorem 2.1.2, if \bar{r} is modular of some Serre weight b , and F_b is a Jordan–Hölder factor of P_a for some Serre weight a , then $a \in W^{\mathrm{cris}}(\bar{r})$. It is easy to find examples of a, b for which $b \in W^{\mathrm{explicit}}(\bar{r})$ but $a \notin W^{\mathrm{explicit}}(\bar{r})$. On the other hand, as explained in [EGHS14] we believe that $W^{\mathrm{cris}}(\bar{r})$ is determined by $W^{\mathrm{explicit}}(\bar{r})$ and a simple combinatorial recipe, so that the weights in $W^{\mathrm{explicit}}(\bar{r})$ are in some sense fundamental.

4.2. Fontaine–Laffaille theory. In applications of our results it is often useful to have information in the opposite direction; namely one wishes to have information about $\bar{r}|_{G_{F_w}}$ at places $v|p$, given that \bar{r} is modular of some particular weight. In the case that l is unramified in F and the weight is sufficiently far inside the lowest alcove, this can be done by Fontaine–Laffaille theory. Again, we specialise to the case that l splits completely in F .

Lemma 4.2.1. Let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, and that l splits completely in F . If n is even, assume that $[F^+ : \mathbb{Q}]n/2$ is even. Suppose that $l > 2$, and that $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_+^n)_0^{\coprod_{w|l} \mathrm{Hom}(k_w, \bar{\mathbb{F}}_l)}$ be a Serre weight. If \bar{r} is modular of weight a , and $w|l$ is such that $a_{w,1} - a_{w,n} \leq l - n$, then $a_w \in W_I^{\mathrm{explicit}}(\bar{r}|_{G_{F_w}}^{\mathrm{ss}})$.

Proof. This is a standard application of Fontaine–Laffaille theory. By Corollary 4.1.8, $\bar{r}|_{G_{F_w}}$ has a crystalline lift with Hodge–Tate weights $a_{w,1} + n - 1, \dots, a_{w,n}$. Since by assumption we have $a_{w,1} + n - 1 - a_{w,n} \leq l - 1$, the result follows immediately from, for example, Proposition 3 of [Wor02] (note that while this reference assumes that the crystalline representation has \mathbb{Q}_l -coefficients, the proof goes through unchanged with $\bar{\mathbb{Q}}_l$ -coefficients). \square

5. EXPLICIT RESULTS FOR GL_3

5.1. We now show how one can obtain cleaner results in the case $n = 3$, making use of the fact that the representation theory of GL_3 , while more complicated than that of GL_2 , is rather simpler than that of GL_n for $n \geq 4$. The following Lemmas are key to our approach.

Lemma 5.1.1. *Let $a \in \mathbb{Z}_+^3$ be a Serre weight for $\mathrm{GL}_3(\mathbb{F}_l)$. Then*

- (1) *if $l - 1 \leq a_1 - a_3$ and $a_1 - a_2, a_2 - a_3 \leq l - 2$, then there is a short exact sequence*

$$0 \rightarrow F_a \rightarrow P_a \rightarrow F_b \rightarrow 0$$

where $b = (a_3 + l - 2, a_2, a_1 - l + 2)$.

- (2) *In all other cases, $P_a = F_a$.*

Proof. This is Proposition 3.18 of [Her09]. \square

Lemma 5.1.2. *Suppose that $n = 3$, and that $a \in \mathbb{Z}_+^3$ is a Serre weight for $\mathrm{GL}_3(\mathbb{F}_l)$. If $a \in W^{\mathrm{explicit}}(\bar{r})$ for some representation $\bar{r} : G_{\mathbb{Q}_l} \rightarrow \mathrm{GL}_3(\mathbb{F}_l)$, then either $a_1 - a_3 = l - 1$ and*

$$\bar{r}|_{I_{\mathbb{Q}_l}} \cong \omega^{-(a_1+1)} \oplus \omega^{-(a_2+1)} \oplus \omega^{-(a_3+1)},$$

or there is a permutation x, y, z of $-(a_1 + 2), -(a_2 + 1), -a_3$ such that $\bar{r}|_{I_{\mathbb{Q}_l}}$ is isomorphic to one of

$$\begin{aligned} & \omega^x \oplus \omega^y \oplus \omega^z, \\ & \omega^x \oplus \omega_2^{y+lz} \oplus \omega_2^{ly+z}, \\ & \omega_3^{x+ly+l^2z} \oplus \omega_3^{y+lz+l^2x} \oplus \omega_3^{z+lx+l^2y}, \end{aligned}$$

where in the second case we have $(l + 1) \nmid ly + z$, and in the third case we have $(l^2 + l + 1) \nmid x + ly + l^2z$.

Proof. This is a simple calculation (it is immediate from the definition that $\bar{r}|_{I_{\mathbb{Q}_l}}$ is of the given form if one ignores the divisibility condition, so the only thing to check is when it can be the case that $ly + z$ is divisible by $l + 1$ or $x + ly + l^2z$ is divisible by $l^2 + l + 1$). \square

Definition 5.1.3. Let $a \in \mathbb{Z}_+^3$ be a Serre weight for $\mathrm{GL}_3(\mathbb{F}_l)$. Then we say that a is *non-generic* if one of the following three conditions hold: $a_1 - a_3 = l - 1$ and $a_1 - a_2, a_2 - a_3 \leq l - 2$; or $a_2 - a_3 = l - 2$ and $a_1 - a_2 \geq 2$; or $a_1 - a_2 = l - 2$ and $a_2 - a_3 \geq 2$. Otherwise we say that a is *generic*.

If l splits completely in F and $a \in (\mathbb{Z}_+^3)_0^{\mathrm{Hom}(F, \overline{\mathbb{Q}_l})}$ is a Serre weight, we say that a is *generic* if for each $\tau \in \mathrm{Hom}(F, \overline{\mathbb{Q}_l})$ the corresponding Serre weight $a_\tau \in \mathbb{Z}_+^3$ is generic.

We remark that this definition of generic is very mild; in particular, it is much less restrictive than the notion of generic used in [EGH13]. (See also Remark 5.1.5 below.)

Theorem 5.1.4. *Let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, and that l splits completely in F . Suppose that $l > 2$, and that $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}_l})$ is an irreducible representation with split ramification. Assume that*

- (1) *There is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_F)$ of weight $\mu \in (\mathbb{Z}_+^3)_0^{\mathrm{Hom}(F, \mathbb{C})}$ and level prime to l such that*

- $\bar{r} \cong \bar{r}_{l,i}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^{\vee} \bar{\epsilon}_l^{-2}$).
- For each $\tau \in \text{Hom}(F, \mathbb{C})$, $\mu_{\tau,1} - \mu_{\tau,3} \leq l - 3$.
- $\bar{r}(G_{F(\zeta_l)})$ is adequate.

Let $a \in (\mathbb{Z}_+^3)_0^{\coprod_{w|l} \text{Hom}(k_w, \bar{\mathbb{F}}_l)}$ be a generic Serre weight. Assume that $a \in W^{\text{explicit}}(\bar{r})$ (so in particular, $\bar{r}|_{G_{F_w}}$ is semisimple for all $w|l$). Then \bar{r} is modular of weight a .

Remark 5.1.5. In fact, the proof below shows that it suffices to assume that a_w is generic for all places $w|l$ for which $\bar{r}|_{G_{F_w}}$ has niveau 2, and that if $\bar{r}|_{G_{F_w}}$ has niveau 1, then we do not have both $a_1 - a_3 = l - 1$ and $a_1 - a_2, a_2 - a_3 \leq l - 2$. In particular, if $\bar{r}|_{G_{F_w}}$ is irreducible for all places $w|l$ (which is the situation considered in [EGH13]), then we do not need to assume that a is generic.

Proof of Theorem 5.1.4. By Corollaries 4.1.10 and 4.1.14, \bar{r} is modular of weight b for some Serre weight b with the property that F_b is a Jordan-Hölder factor of P_a . We wish to show that $F_b \cong F_a$. Assume for the sake of contradiction that $F_b \not\cong F_a$, so that there is a place $w|l$ with $F_{b_w} \not\cong F_{a_w}$. By Lemma 5.1.1, we must have $l - 1 \leq a_{w,1} - a_{w,3}$ and $a_{w,1} - a_{w,2}, a_{w,2} - a_{w,3} \leq l - 2$, and $b_w = (a_{w,3} + l - 2, a_{w,2}, a_{w,1} - l + 2)$.

Since $l - 1 \leq a_{w,1} - a_{w,3}$, we have $b_{w,1} - b_{w,3} = 2l - 4 - (a_{w,1} - a_{w,3}) \leq l - 3$. Thus the assumption that \bar{r} is modular of weight b , together with Lemma 4.2.1 gives an explicit description of the possibilities for $\bar{r}|_{G_{F_w}}$ (which is assumed to be semisimple) in terms of b_w , and hence in terms of a_w . We also have another such description from the assumption that $a \in W^{\text{explicit}}(\bar{r})$. We will now compare these descriptions to obtain a contradiction.

It will be useful to note that since we are assuming that $a_{w,1} - a_{w,2}, a_{w,2} - a_{w,3} \leq l - 2$, and $a_{w,1} - a_{w,3} \geq l - 1$ we have

$$(5.1.1) \quad 1 \leq a_{w,1} - a_{w,2}, a_{w,2} - a_{w,3} \leq l - 2,$$

$$(5.1.2) \quad l - 1 \leq a_{w,1} - a_{w,3} \leq 2l - 4,$$

so that

$$(5.1.3) \quad a_{w,1} \not\equiv a_{w,2} \pmod{l - 1},$$

$$(5.1.4) \quad a_{w,2} \not\equiv a_{w,3} \pmod{l - 1},$$

$$(5.1.5) \quad a_{w,3} \not\equiv a_{w,1} + 1 \pmod{l - 1}.$$

$$(5.1.6) \quad a_{w,1} - a_{w,3} \not\equiv l - 2 \pmod{l + 1}.$$

If $a_{w,1} - a_{w,2} = 1$ then the condition that $a_{w,1} - a_{w,3} \geq l - 1$ forces $a_{w,2} - a_{w,3} = l - 2$, so that a_w is not generic. Similarly if $a_{w,2} - a_{w,3} = 1$ then a_w is not generic. Therefore if we assume that a_w is generic, we also have

$$(5.1.7) \quad a_{w,1} \not\equiv a_{w,2} + 1 \pmod{l - 1},$$

$$(5.1.8) \quad a_{w,2} \not\equiv a_{w,3} + 1 \pmod{l - 1}.$$

By the second and third conditions in the definition of genericity, we also have

$$(5.1.9) \quad a_{w,3} \not\equiv a_{w,2} + 1 \pmod{l - 1},$$

$$(5.1.10) \quad a_{w,2} \not\equiv a_{w,1} + 1 \pmod{l - 1}.$$

Niveau 1 Suppose firstly that $\bar{r}|_{G_{F_w}}$ has niveau 1, i.e. that $\bar{r}|_{I_{F_w}}$ is a direct sum of powers of the mod l cyclotomic character ω . Then since $a \in W^{\text{explicit}}(\bar{r})$ and a is generic, we see from Lemma 5.1.2 that

$$\bar{r}|_{I_{F_w}} \cong \omega^{-(a_{w,1}+2)} \oplus \omega^{-(a_{w,2}+1)} \oplus \omega^{-a_{w,3}}.$$

By Lemma 4.2.1 (applied to F_b), we see that we also have

$$\bar{r}|_{I_{F_w}} \cong \omega^{-(a_{w,3}+1)} \oplus \omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,1}+1)}.$$

Thus $a_{w,3} \equiv a_{w,1} + 1 \pmod{l-1}$, contradicting (5.1.5).

Niveau 2 Suppose next that $\bar{r}|_{G_{F_w}}$ has niveau 2, i.e. that $\bar{r}|_{I_{F_w}}$ is a direct sum of a power of the mod l cyclotomic character ω and characters $\omega_2^n, \omega_2^{ln}$ for some n with $(l+1) \nmid n$, where ω_2 is a choice of fundamental character of niveau 2. Then since $a \in W^{\text{explicit}}(\bar{r})$, we see from Lemma 5.1.2 that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\begin{aligned} & \omega^{-(a_{w,1}+2)} \oplus \omega_2^{-(a_{w,2}+1+la_{w,3})} \oplus \omega_2^{-(l(a_{w,2}+1)+a_{w,3})} \\ & \omega^{-(a_{w,2}+1)} \oplus \omega_2^{-(a_{w,1}+2+la_{w,3})} \oplus \omega_2^{-(l(a_{w,1}+2)+a_{w,3})} \\ & \omega^{-a_{w,3}} \oplus \omega_2^{-(a_{w,1}+2+l(a_{w,2}+1))} \oplus \omega_2^{-(l(a_{w,1}+2)+a_{w,2}+1)} \end{aligned}$$

By Lemma 4.2.1 (applied to F_b), we see that we also have that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\begin{aligned} & \omega^{-(a_{w,1}+1)} \oplus \omega_2^{-(a_{w,2}+1+l(a_{w,3}+l))} \oplus \omega_2^{-(l(a_{w,2}+1)+a_{w,3}+l)} \\ & \omega^{-(a_{w,2}+1)} \oplus \omega_2^{-(a_{w,1}-l+2+l(a_{w,3}+l))} \oplus \omega_2^{-(l(a_{w,1}-l+2)+(a_{w,3}+l))} \\ & \omega^{-(a_{w,3}+1)} \oplus \omega_2^{-(a_{w,1}-l+2+l(a_{w,2}+1))} \oplus \omega_2^{-(l(a_{w,1}-l+2)+a_{w,2}+1)} \end{aligned}$$

Comparing the powers of ω and using (5.1.3)–(5.1.10), the only possibility is that we simultaneously have

$$\begin{aligned} \bar{r}|_{G_{F_w}} & \cong \omega^{-(a_{w,2}+1)} \oplus \omega_2^{-(a_{w,1}+2+la_{w,3})} \oplus \omega_2^{-(l(a_{w,1}+2)+a_{w,3})}, \\ \bar{r}|_{G_{F_w}} & \cong \omega^{-(a_{w,2}+1)} \oplus \omega_2^{-(a_{w,1}-l+2+l(a_{w,3}+l))} \oplus \omega_2^{-(l(a_{w,1}-l+2)+(a_{w,3}+l))}. \end{aligned}$$

There are now two possibilities to examine. Firstly it could be the case that

$$a_{w,1} + 2 + la_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) \pmod{l^2 - 1};$$

but this implies that $l^2 - l \equiv 0 \pmod{l^2 - 1}$, a contradiction. So we must have

$$a_{w,1} + 2 + la_{w,3} \equiv l(a_{w,1} - l + 2) + (a_{w,3} + l) \pmod{l^2 - 1}.$$

This simplifies to $a_{w,1} - a_{w,3} \equiv l - 2 \pmod{l+1}$, contradicting (5.1.6).

Niveau 3 Suppose finally that $\bar{r}|_{G_{F_w}}$ has niveau 3, i.e. that $\bar{r}|_{I_{F_w}}$ is of the form $\omega_3^n \oplus \omega_3^{ln} \oplus \omega_3^{l^2n}$ for some n with $(l^2 + l + 1) \nmid n$, where ω_3 is a choice of fundamental character of niveau 3. Then since $a \in W^{\text{explicit}}(\bar{r})$, we see that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\begin{aligned} & \omega_3^{-(a_{w,1}+2+l(a_{w,2}+1)+l^2a_{w,3})} \oplus \omega_3^{-(a_{w,2}+1+la_{w,3}+l^2(a_{w,1}+2))} \oplus \omega_3^{-(a_{w,3}+l(a_{w,1}+2)+l^2(a_{w,2}+1))} \\ & \omega_3^{-(a_{w,1}+2+la_{w,3}+l^2(a_{w,2}+1))} \oplus \omega_3^{-(a_{w,3}+l(a_{w,2}+1)+l^2(a_{w,1}+2))} \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,1}+2)+l^2a_{w,3})} \end{aligned}$$

On the other hand, by Lemma 4.2.1 (applied to F_b) we also have that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\omega_3^{-(a_{w,1}-l+2+l(a_{w,2}+1)+l^2(a_{w,3}+l))} \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,3}+l)+l^2(a_{w,1}-l+2))} \oplus \omega_3^{-(a_{w,3}+l(a_{w,1}-l+2)+l^2(a_{w,2}+1))}$$

$$\omega_3^{-(a_{w,1}-l+2+l(a_{w,3}+l)+l^2(a_{w,2}+1))} \oplus \omega_3^{-(a_{w,3}+l+l(a_{w,2}+1)+l^2(a_{w,1}-l+2))} \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,1}-l+2)+l^2(a_{w,3}+l))}$$

Examining the exponents in these expressions, we obtain 12 possible congruences (mod $l^3 - 1$), each of which we will now show yields a contradiction. In each case below we derive a congruence modulo $l^2 + l + 1$ or $l^3 - 1$, and it is easy to see in each case that the inequalities (5.1.1) and (5.1.2) imply that the congruence has no solutions.

- (1) $a_{w,1} + 2 + l(a_{w,2} + 1) + l^2 a_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,2} + 1) + l^2(a_{w,3} + l) \pmod{l^3 - 1}$. This simplifies to $l^2 - 1 \equiv 0 \pmod{l^3 - 1}$, a contradiction.
- (2) $a_{w,1} + 2 + l(a_{w,2} + 1) + l^2 a_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}$. This simplifies to $a_{w,2} - a_{w,3} + 2 \equiv 0 \pmod{l^2 + l + 1}$, a contradiction.
- (3) $a_{w,1} + 2 + l(a_{w,2} + 1) + l^2 a_{w,3} \equiv a_{w,2} + 1 + l(a_{w,1} - l + 2) + l^2(a_{w,3} + l) \pmod{l^3 - 1}$. This simplifies to $a_{w,1} - a_{w,2} \equiv l \pmod{l^2 + l + 1}$, a contradiction.
- (4) $a_{w,1} + 2 + l(a_{w,2} + 1) + l^2 a_{w,3} \equiv a_{w,2} + 1 + l(a_{w,3} + l) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}$. This simplifies to $l(a_{w,1} - a_{w,3} + 3) + (a_{w,1} - a_{w,2} + 2) \equiv 0 \pmod{l^2 + l + 1}$, which is easily seen to be impossible.
- (5) $a_{w,1} + 2 + l(a_{w,2} + 1) + l^2 a_{w,3} \equiv a_{w,3} + l + l(a_{w,1} - l + 2) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}$. This simplifies to $(a_{w,1} - a_{w,3}) + l(a_{w,2} - a_{w,3}) + 2 \equiv 0 \pmod{l^2 + l + 1}$, which is also impossible.
- (6) $a_{w,1} + 2 + l(a_{w,2} + 1) + l^2 a_{w,3} \equiv a_{w,3} + l + l(a_{w,2} + 1) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}$. This simplifies to $(l+1)(a_{w,1} - a_{w,3} + 2) + 1 \equiv 0 \pmod{l^2 + l + 1}$, which is impossible.
- (7) $a_{w,1} + 2 + l a_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,1} - l + 2 + l(a_{w,2} + 1) + l^2(a_{w,3} + l) \pmod{l^3 - 1}$. This simplifies to $l(a_{w,2} - a_{w,3} + 1) + 1 \equiv 0 \pmod{l^2 + l + 1}$, a contradiction.
- (8) $a_{w,1} + 2 + l a_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}$. This simplifies to $l^2 - l \equiv 0 \pmod{l^3 - 1}$, a contradiction.
- (9) $a_{w,1} + 2 + l a_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,2} + 1 + l(a_{w,1} - l + 2) + l^2(a_{w,3} + l) \pmod{l^3 - 1}$. This simplifies to $l(a_{w,2} - a_{w,3} + 2) \equiv a_{w,1} - a_{w,2} \pmod{l^2 + l + 1}$, which is easily seen to be impossible.
- (10) $a_{w,1} + 2 + l a_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,2} + 1 + l(a_{w,3} + l) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}$. This simplifies to $a_{w,1} - a_{w,2} + 2 \equiv 0 \pmod{l^2 + l + 1}$, a contradiction.
- (11) $a_{w,1} + 2 + l a_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,3} + l + l(a_{w,1} - l + 2) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}$. This simplifies to $a_{w,1} - a_{w,3} \equiv l - 2 \pmod{l^2 + l + 1}$, a contradiction.
- (12) $a_{w,1} + 2 + l a_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,3} + l + l(a_{w,2} + 1) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}$. This simplifies to $l(a_{w,1} - a_{w,2} + 1) + a_{w,1} - a_{w,3} + 3 \equiv 0 \pmod{l^2 + l + 1}$, which is impossible.

As we have obtained a contradiction in every case, we see that $F_b \cong F_a$, as required. \square

REFERENCES

- [ADP02] Avner Ash, Darrin Doud, and David Pollack, *Galois representations with conjectural connections to arithmetic cohomology*, Duke Math. J. **112** (2002), no. 3, 521–579.

- [BC11] Joël Bellaïche and Gaëtan Chenevier, *The sign of Galois representations attached to automorphic forms for unitary groups*, Compos. Math. **147** (2011), no. 5, 1337–1352.
- [BLGG11] Tom Barnet-Lamb, Toby Gee, and David Geraghty, *The Sato-Tate Conjecture for Hilbert Modular Forms*, J. Amer. Math. Soc. **24** (2011), no. 2, 411–469.
- [BLGG12] Thomas Barnet-Lamb, Toby Gee, and David Geraghty, *Congruences between Hilbert modular forms: constructing ordinary lifts*, Duke Mathematical Journal **161** (2012), no. 8, 1521–1580.
- [BLGG13] ———, *Serre weights for rank two unitary groups*, Math. Ann. **356** (2013), no. 4, 1551–1598.
- [BLGGT14a] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Local-global compatibility for $l = p$, II.*, Ann. Sci. École Norm. Sup. **47** (2014), no. 1, 161–175.
- [BLGGT14b] Tom Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (to appear) (2014).
- [Car12a] Ana Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413.
- [Car12b] ———, *Local-global compatibility for $l \neq p$* , preprint arXiv:1202.4683, 2012.
- [CH13] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations, II*, Cambridge Journal of Mathematics **1** (2013), 57–74.
- [CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l -adic lifts of automorphic mod l Galois representations*, Pub. Math. IHES **108** (2008), 1–181.
- [EGH13] Matthew Emerton, Toby Gee, and Florian Herzig, *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), no. 9, 1649–1722.
- [EGHS14] Matthew Emerton, Toby Gee, Florian Herzig, and David Savitt, *Explicit Serre weight conjectures*, in preparation, 2014.
- [Gee06] Toby Gee, *A modularity lifting theorem for weight two Hilbert modular forms*, Math. Res. Lett. **13** (2006), no. 5–6, 805–811.
- [Gee11] ———, *Automorphic lifts of prescribed types*, Math. Ann. **350** (2011), no. 1, 107–144.
- [GG12] Toby Gee and David Geraghty, *Companion forms for unitary and symplectic groups*, Duke Math. J. **161** (2012), no. 2, 247–303.
- [GHTT10] Robert Guralnick, Florian Herzig, Richard Taylor, and Jack Thorne, *Adequate subgroups*, Appendix to [Tho12], 2010.
- [GL12] Hui Gao and Tong Liu, *A note on potential diagonalizability of crystalline representations*, 2012.
- [GLS13] Toby Gee, Tong Liu, and David Savitt, *The weight part of Serre’s conjecture for $GL(2)$* , 2013.
- [GLS14] ———, *The Buzzard–Diamond–Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. (to appear).
- [GS11] Toby Gee and David Savitt, *Serre weights for mod p Hilbert modular forms: the totally ramified case*, J. Reine Angew. Math. **660** (2011), 1–26.
- [Her09] Florian Herzig, *The weight in a Serre-type conjecture for tame n -dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116.
- [HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.
- [Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.
- [Kis08] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546.
- [Kis09] ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180.
- [Lab09] Jean-Pierre Labesse, *Changement de base CM et séries discrètes*, preprint, 2009.
- [Shi11] Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741.

- [Tho12] Jack Thorne, *On the automorphy of l -adic Galois representations with small residual image*, J. Inst. Math. Jussieu **11** (2012), no. 4, 855–920, With an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne.
- [TY07] Richard Taylor and Teruyoshi Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493 (electronic).
- [Wor02] Sigrid Wortmann, *Galois representations of three-dimensional orthogonal motives*, Manuscripta Math. **109** (2002), no. 1, 1–28.

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